Backward error of polynomial eigenvalue problems solved by linearization

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It is commonplace in many application domains to utilize polynomial eigenvalue problems to model the behaviour of physical systems. Many techniques exist to compute solutions of these polynomial eigenvalue problems. One of the most frequently used techniques is linearization, in which the polynomial eigenvalue problem is turned into an equivalent linear eigenvalue problem with the same eigenvalues, and with easily recoverable eigenvectors. The eigenvalues and eigenvectors of the linearization are usually computed using a backward stable solver such as the QZ algorithm. Such backward stable algorithms ensure that the computed eigenvalues and eigenvectors of the linearization are exactly those of a nearby linear pencil, where the perturbations are bounded in terms of the machine precision and the norms of the matrices defining the linearization. With respect to the linearization, we may have solved a nearby problem, but we would also like to know if our computed solution is the exact solution of a nearby polynomial eigenvalue problem.

Furthermore, there has recently been a steady increase in the number of distinct linearizations proposed in the literature, depending mainly on the basis in which the polynomial eigenvalue problems are represented. Certainly, the choice of basis can have a dramatic effect on the backward errors, as can the particular choice of linearization. One of the objectives of this work is to develop a framework for analyzing different polynomial bases and linearizations in a uniform way. Thus, we investigate a particular class of linearizations where the polynomial coefficients are separated from the recurrence relations of the polynomial basis employed.

We use one-sided factorization to relate the linearization to the original polynomial in a very particular way. Given a linearization $\mathcal{L}(\lambda)$ of a polynomial matrix $P(\lambda)$, we find a one-sided factorization $\Phi(\lambda)$, such that $\mathcal{L}(\lambda)\Phi(\lambda) = P(\lambda) \otimes e_1$, where e_1 is the first unit vector. Since the QZ algorithm computes the exact solution of a slightly perturbed linearization, we investigate

$$(\mathcal{L}(\lambda) + \Delta\mathcal{L}(\lambda))(\Phi(\lambda) + \Delta\Phi(\lambda)) = \begin{bmatrix} P(\lambda) + \Delta P(\lambda) \\ 0 \end{bmatrix}, \tag{1}$$

to first order. The perturbation $\Delta\Phi(\lambda)$ to the one-sided factorization is chosen in order to maintain the structure in the bottom of (1). For a given specific basis, we utilize the appropriate convolution matrices in order to obtain upper bounds for the norm of the coefficients of the perturbation $\Delta P(\lambda)$.

For some specific polynomial bases and linearizations, we are able to formulate these upper bounds in a simple way. Thus, we obtain the conditions under which the backward error of the solution of the polynomial eigenvalue problems are small.