Linearizations for Interpolation Bases

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A standard approach to solving the polynomial eigenvalue problem is to linearize, which is to say the problem is transformed into an equivalent larger order generalized eigenproblem. For the monomial basis, much work has been done to show the conditions under which linearizations produce small backward errors, especially for the quadratic eigenvalue problem [3, 4]. Recently, there has been increasing interest in linearizations of polynomials expressed in bases other than the classical monomial basis [1]. In these bases, there is a need to establish the conditions under which linearizations return eigenvalue and eigenvector estimates with small backward errors.

In this work, we investigate the accuracy and stability of polynomial eigenvalue problems solved by linearization. The polynomial eigenvalue problems are expressed in the Lagrange basis, that is, by their values at distinct interpolation nodes. We also utilize the barycentric Lagrange formulation of the polynomial matrices, since the linearizations that arise from this formulation are particularly simple, and are flexible for computations. An m by m matrix polynomial $\mathbf{P}(\lambda)$ of degree n, expressed in the barycentric Lagrange formulation is

$$\mathbf{P}(\lambda) = \prod_{i=0}^{n}(\lambda - x_i)\sum_{j=0}^{n}\frac{w_j}{\lambda - x_j}\mathbf{F}_j \,, \quad w_j^{-1} = \prod_{k\neq j}(x_j - x_k)\,.$$

The numbers w_j are known as the barycentric weights, and the coefficients $\mathbf{F}_j = \mathbf{P}(x_j) \in \mathbb{C}^{m\times m}$ are the samples of $\mathbf{P}(\lambda)$ at the $n+1$ interpolation nodes x_j. An $(n+2)m$ by $(n+2)m$ linearization of the matrix polynomial $\mathbf{P}(\lambda)$ is given by [2]

$$\lambda\mathbf{B} - \mathbf{A} = \begin{bmatrix} \mathbf{0}_m & \mathbf{F}_0 & \cdots & \mathbf{F}_n \\ -w_0\mathbf{I}_m & (\lambda - x_0)\mathbf{I}_m & & \\ \vdots & & \ddots & \\ -w_n\mathbf{I}_m & & & (\lambda - x_n)\mathbf{I}_m \end{bmatrix}\,. \tag{1}$$

This linearization introduces $2m$ spurious infinite eigenvalues. However, these spurious eigenvalues are not a problem in practice, and can be deflated from the pencil with little extra effort. The advantage of introducing the spurious eigenvalues comes in the form of flexibility when balancing the linearization. Furthermore, when the interpolation nodes are real or on the unit circle we may perform a reduction to block Hessenberg form in only $O(n^2)$ operations. This process is related to an inverse eigenvalue problem for computing the recurrence coefficients of orthogonal polynomials with respect to a discrete inner product defined by the interpolation nodes [5].

For the linearization (1), we show the conditions under which the backward error of the polynomial eigenvalue problem is not much larger than that of the backward error of the linearization. We also investigate the stability of two smaller linearizations recently proposed for polynomials expressed in barycentric form [6].

References

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