Semi-practical algorithms for computing the periodic points of the Mandelbrot set

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In this talk, we will survey a number of numerical techniques for computing the roots of univariate polynomials. We will explore a particular class of polynomials related to the periodic points of the Mandelbrot set. We are interested in computing the points c in the complex plane for which the iteration

$$z_{n+1} = z_n^2 + c\,,$$

starting from the critical point $z_0 = 0$ returns to zero after k steps, that is, the points c for which $z_k = 0$. These points can be computed as the zeros of the polynomial $p_k(\zeta)$, of degree $2^{k-1} - 1$ for $k \geq 1$, generated from recurrence relation

$$p_{k+1}(\zeta) = \zeta(p_k(\zeta))^2 + 1\,, \quad p_0(\zeta) = 0\,.$$

We use this family of polynomials to explore a variety of numerical algorithms to compute the roots, and in the process we discover some interesting ways of computing polynomials roots.

We describe a family of recursively constructed Hessenberg matrices whose characteristic polynomial is equal to $p_k(\zeta)$. The matrices are extremely sparse, and we can exploit the structure to efficiently compute the roots as eigenvalues of the matrices. We also explore other methodologies such as linearization based approaches and continuation based methods for computing the roots of such polynomials.

If time permits, we will describe a linearization based approach for computing the roots of polynomials related to random unitary matrices. In particular, given a polynomial

$$q(\lambda) = \det\left(\lambda I - U\right)\,,$$

where U is drawn from the circular unitary ensemble, we explore algorithms for computing the zeros of the derivative $q'(\lambda)$ directly from the eigenvalues of U.